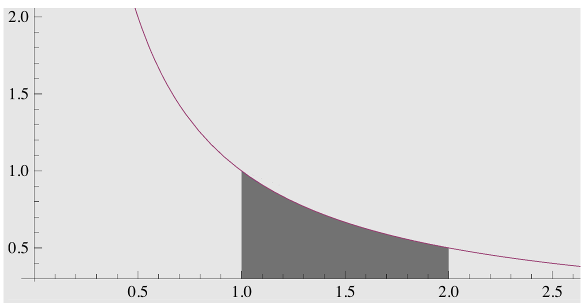
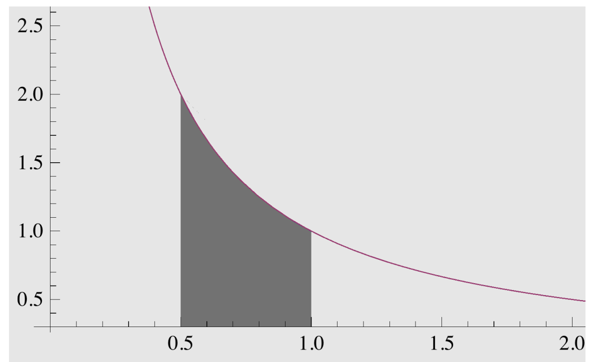
Lecture 2 : The Natural Logarithm.

Recall
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1.$$

What happens if $n = -1$?

Definition We can define a function which is **an anti-derivative for** x^{-1} using the Fundamental Theorem of Calculus: We let
$$\ln x = \int_1^x \frac{1}{t} dt, \qquad x > 0.$$
This function is called the natural logarithm.

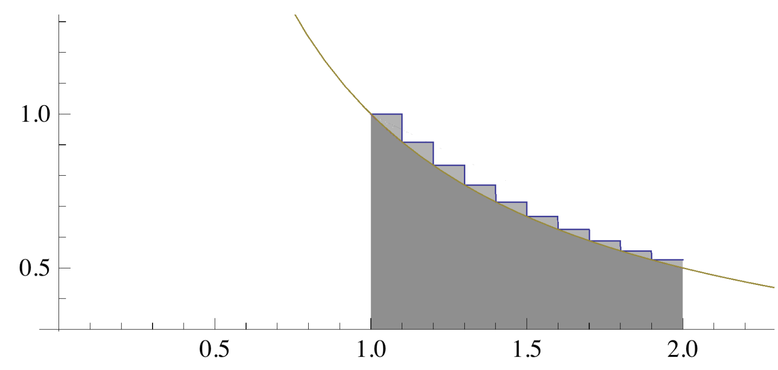




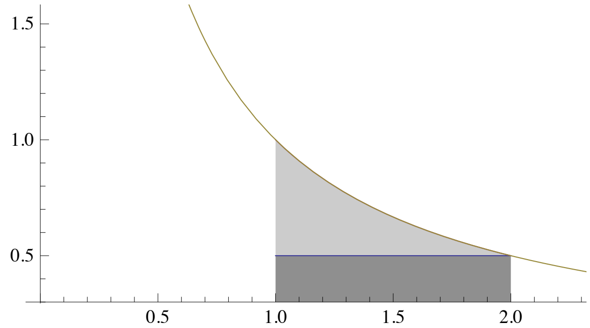
Note that $\ln(x)$ is the area under the continuous curve $y = \frac{1}{t}$ between 1 and x if $x > 1$ and minus the area under the continuous curve $y = \frac{1}{t}$ between 1 and x if $x < 1$.

We have $\ln(2)$ is the area of the region shown in the picture on the left above and $\ln(1/2)$ is minus the area of the region shown in the picture on the right above.

I do not have a formula for $\ln(x)$ in terms of functions studied before, however I could estimate the value of $\ln(2)$ using a Riemann sum. The approximating rectangles for a left Riemann sum with 10 approximating rectangles is shown below. Their area adds to 0.718771 (to 6 decimal places). If we took the limit of such sums as the number of approximating rectangles tends to infinity, we would get the actual value of $\ln(2)$, which is 0.693147 (to 6 decimal places). The natural logarithm function is a vuilt in function on most scientific calculators.



With very little work, using a right Riemann sum with 1 approximating rectangle, we can get a lower bound for $\ln(2)$. The picture below demonstrates that $\boxed{\ln 2 = \int_1^2 \frac{1}{t} dt > 1/2}$.



Properties of the Natural Logarithm:

We can use our tools from Calculus I to derive a lot of information about the natural logarithm.

1. Domain $= (0, \infty)$ (by definition)

2. Range $= (-\infty, \infty)$ (see later)

3. $\ln x > 0$ if $x > 1$, $\ln x = 0$ if $x = 1$, $\ln x < 0$ if $x < 1$.

   This follows from our comments above after the definition about how $ln(x)$ relates to the area under the curve $y = 1/x$ between 1 and x.

4. $\frac{d(\ln x)}{dx} = \frac{1}{x}$

   This follows from the definition and the Fundamental Theorem of Calculus.

5. The graph of $y = \ln x$ is increasing, continuous and concave down on the interval $(0, \infty)$.

   Let $f(x) = \ln(x)$, $f'(x) = 1/x$ which is always positive for $x > 0$ (the domain of f), Therefore the graph of $f(x)$ is increasing on its domain. We have $f''(x) = \frac{-1}{x^2}$ which is always negative, showing that the graph of $f(x)$ is concave down. The function f is continuous since it is differentiable.

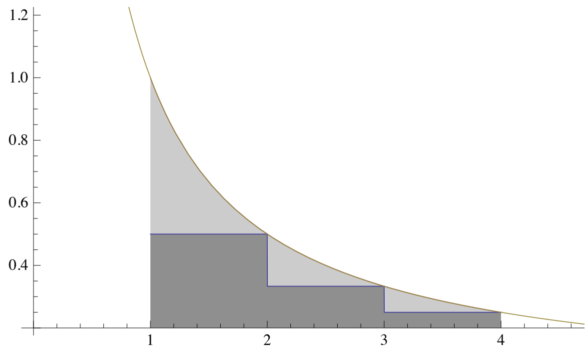
6. The function $f(x) = \ln x$ is a one-to-one function.

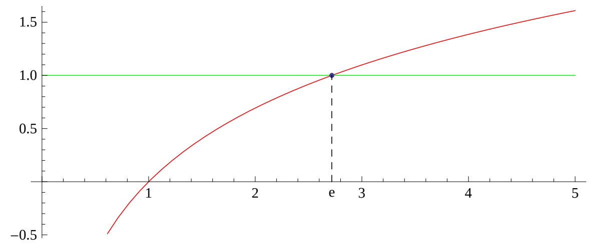
   Since $f'(x) = 1/x$ which is positive on the domain of f, we can conclude that f is a one-to-one function.

7. Since $f(x) = \ln x$ is a one-to-one function, there is a unique number, e, with the property that

$$\boxed{\ln e = 1.}$$

   We have $ln(1) = 0$ since $\int_1^1 1/t \, dt = 0$. Using a Riemann sum with 3 approximating rectangles, we see that $ln(4) > 1/1 + 1/2 + 1/3 > 1$. Therefore by the intermediate value theorem, since $f(x) = \ln(x)$ is continuous, there must be some number e with $1 < e < 4$ for which $\ln(e) = 1$. This number is unique since the function $f(x) = \ln(x)$ is one-to-one.





We will be able to estimate the value of e in the next section with a limit. $e \approx 2.7182818284590$.

The following properties are very useful when calculating with the natural logarithm:

$$\boxed{\begin{array}{c} (i) \quad \ln 1 = 0 \\ \\ (ii) \quad \ln(ab) = \ln a + \ln b \\ \\ (iii) \quad \ln(\frac{a}{b}) = \ln a - \ln b \\ \\ (iv) \quad \ln a^r = r \ln a \end{array}}$$

where a and b are positive numbers and r is a rational number.

Proof (ii) We show that $\ln(ax) = \ln a + \ln x$ for a constant $a > 0$ and any value of $x > 0$. The rule follows with $x = b$. Let $f(x) = \ln x, \quad x > 0$ and $g(x) = \ln(ax), \quad x > 0$. We have $f'(x) = \frac{1}{x}$ and $g'(x) = \frac{1}{ax} \cdot a = \frac{1}{x}$.

Since both functions have equal derivatives, $f(x) + C = g(x)$ for some constant C. Substituting $x = 1$ in this equation, we get $\ln 1 + C = \ln a$, giving us $C = \ln a$ and $\ln ax = \ln a + \ln x$.

(iii) Note that $0 = \ln 1 = \ln \frac{a}{a} = \ln a \cdot \frac{1}{a} = \ln a + \ln \frac{1}{a}$, giving us that $\ln \frac{1}{a} = -\ln a$.
Thus we get $\ln \frac{a}{b} = \ln a + \ln \frac{1}{b} = \ln a - \ln b$.

(iv) Comparing derivatives, we see that

$$\frac{d(\ln x^r)}{dx} = \frac{rx^{r-1}}{x^r} = \frac{r}{x} = \frac{d(r \ln x)}{dx}.$$

Hence $\ln x^r = r \ln x + C$ for any $x > 0$ and any rational number r. Letting $x = 1$ we get $C = 0$ and the result holds.

Example Expand

$$\ln \frac{x^2\sqrt{x^2+1}}{x^3}$$

using the rules of logarithms.

Example Express as a single logarithm:

$$\ln x + 3\ln(x+1) - \frac{1}{2}\ln(x+1).$$

Example Evaluate $\int_1^{e^2} \frac{1}{t} dt$

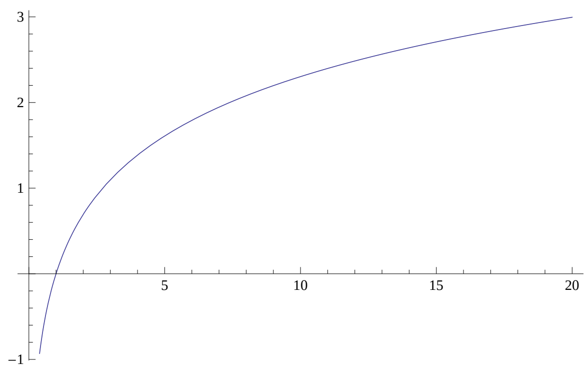
We can use the rules of logarithms given above to derive the following information about limits.

$$\boxed{\lim_{x\to\infty} \ln x = \infty, \qquad \lim_{x\to 0} \ln x = -\infty.}$$

Proof We saw above that $\ln 2 > 1/2$. If $x > 2^n$, then $\ln x > \ln 2^n$ (Why ?). So $\ln x > n \ln 2 > n/2$. Hence as $x \to \infty$, the values of $\ln x$ also approach ∞.

Also $\ln \frac{1}{2^n} = -n \ln 2 < -n/2$. Thus as x approaches 0 the values of $\ln x$ approach $-\infty$.

Note that we can now draw a reasonable sketch of the graph of $y = \ln(x)$, using all of the information derived above.

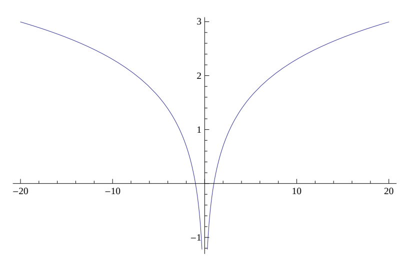


Example Find the limit $\lim_{x\to\infty} \ln(\frac{1}{x^2+1})$.

We can extend the applications of the natural logarithm function by composing it with the absolute value function. We have :

$$\ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

This is an even function with graph



We have $ln|x|$ is also an antiderivative of $1/x$ with a larger domain than $ln(x)$.

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}} \quad \text{and} \quad \boxed{\int \frac{1}{x}dx = \ln|x| + C}$$

We can use the chain rule and integration by substitution to get

$$\boxed{\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}} \quad \text{and} \quad \boxed{\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C}$$

Example Differentiate $\ln|\sqrt[3]{x-1}|$.

Example Find the integral

$$\int \frac{x}{3-x^2}dx.$$

Logarithmic Differentiation

To differentiate $y = f(x)$, it is often easier to use logarithmic differentiation :

1. Take the natural logarithm of both sides to get $\ln y = \ln(f(x))$.
2. Differentiate with respect to x to get $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln(f(x))$
3. We get $\frac{dy}{dx} = y\frac{d}{dx}\ln(f(x)) = f(x)\frac{d}{dx}\ln(f(x))$.

Example Find the derivative of $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$.

Extra Examples

Please try to work through these questions before looking at the solutions.

Example Expand $\ln(\frac{e^2\sqrt{a^2+1}}{b^3})$

Example Differentiate $\ln|\sqrt[3]{x-1}|$.

Example Find $d/dx \ln(|\cos x|)$.

Example Find the integral

$$\int \cot x dx$$

Example Find the integral

$$\int_e^{e^2} \frac{1}{x \ln x} dx.$$

Example Find the derivative of $y = \frac{\sin^2 x \tan^4 x}{(x^2-1)^2}$.

Old Exam Question Differentiate the function

$$f(x) = \frac{(x^2-1)^4}{\sqrt{x^2+1}}.$$

Solutions

Example Expand $\ln(\frac{e^2\sqrt{a^2+1}}{b^3})$

$$\ln(\frac{e^2\sqrt{a^2+1}}{b^3}) = \ln(e^2\sqrt{a^2+1}) - \ln(b^3) = \ln(e^2) + \ln(\sqrt{a^2+1}) - 3\ln b$$
$$= 2\ln e + \frac{1}{2}\ln(a^2+1) - 3\ln b = 2 + \frac{1}{2}\ln(a^2+1) - 3\ln b.$$

Example Differentiate $\ln|\sqrt[3]{x-1}|$.

We use the chain rule here

$$\frac{d}{dx}\ln|\sqrt[3]{x-1}| = \frac{1}{\sqrt[3]{x-1}} \cdot \frac{1}{3}(x-1)^{-2/3} = \frac{1}{3(x-1)}.$$

Example Find $d/dx \ln(|\cos x|)$.

Again, we use the chain rule

$$\frac{d}{dx}\ln|\cos x| = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x.$$

Example Find the integral

$$\int \cot x dx$$

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx.$$

We use substitution. Let $u = \sin x$, $du = \cos x dx$.

$$\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$$

Example Find the integral

$$\int_e^{e^2} \frac{1}{x\ln x} dx.$$

We use substitution. Let $u = \ln x$, $du = \frac{1}{x}dx$. $u(e) = \ln e = 1$, $u(e^2) = \ln e^2 = 2$.

$$\int_e^{e^2} \frac{1}{x\ln x} dx = \int_1^2 \frac{du}{u} = \ln|u|\Big|_1^2 = \ln 2 - \ln 1 = \ln 2.$$

Example Find the derivative of $y = \frac{\sin^2 x \tan^4 x}{(x^2-1)^2}$.

We use Logarithmic differentiation. If $y = \frac{\sin^2 x \tan^4 x}{(x^2-1)^2}$, then

$$\ln y = \ln(\sin^2 x) + \ln(\tan^4 x) - \ln((x^2-1)^2) = 2\ln(\sin x) + 4\ln(\tan x) - 2\ln(x^2-1).$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\cos x}{\sin x} + \frac{4\sec^2 x}{\tan x} - \frac{2(2x)}{x^2-1}.$$

Multiplying both sides by y and converting to a function of x, we get

$$\frac{dy}{dx} = y\Big[\frac{2\cos x}{\sin x} + \frac{4\sec^2 x}{\tan x} - \frac{4x}{x^2-1}\Big] = \Big(\frac{\sin^2 x \tan^4 x}{(x^2-1)^2}\Big)\Big[\frac{2\cos x}{\sin x} + \frac{4\sec^2 x}{\tan x} - \frac{4x}{x^2-1}\Big].$$

Old Exam Question Differentiate the function

$$f(x) = \frac{(x^2-1)^4}{\sqrt{x^2+1}}.$$

We use Logarithmic differentiation. If $y = \frac{(x^2-1)^4}{\sqrt{x^2+1}}$, then

$$\ln y = 4\ln(x^2-1) - \frac{1}{2}\ln(x^2+1).$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{4(2x)}{x^2-1} - \frac{2x}{2(x^2+1)}.$$

Multiplying both sides by y and converting to a function of x, we get

$$\frac{dy}{dx} = y\Big[\frac{8x}{x^2-1} - \frac{x}{(x^2+1)}\Big] = \Big(\frac{(x^2-1)^4}{\sqrt{x^2+1}}\Big)\Big[\frac{8x}{x^2-1} - \frac{x}{(x^2+1)}\Big].$$